

STANDARD RESULTS IN TRIGONOMETRY

Section - 3

3.1. Trigonometric Ratios for sum and difference of angles :

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$$\begin{aligned} \text{➤ } \tan(A + B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \left[\text{where } A \neq n\pi + \frac{\pi}{2}, B \neq n\pi + \frac{\pi}{2} \right] \\ \tan(A - B) &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \left[\text{and } A \pm B \neq m\pi + \frac{\pi}{2} \right] \end{aligned}$$

$$\begin{aligned} \text{➤ } \cot(A + B) &= \frac{\cot A \cdot \cot B - 1}{\cot A + \cot B} \left[\text{where } A \neq n\pi, B \neq n\pi \right] \\ \cot(A - B) &= \frac{\cot A \cdot \cot B + 1}{\cot B - \cot A} \left[\text{and } A \pm B \neq m\pi \right] \end{aligned}$$

$$\text{➤ } \tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$\text{➤ } \cot(A + B + C) = \frac{\cot A + \cot B + \cot C - \cot A \cot B \cot C}{1 - \cot A \cot B - \cot B \cot C - \cot C \cot A}$$

$$\text{➤ } \sin(A + B + C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$$

or,

$$\sin(A + B + C) = \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C)$$

$$\text{➤ } \cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$$

or,

$$\cos(A + B + C) = \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$$

$$\text{➤ } \sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$

$$\text{➤ } \cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$$

$$\text{➤ } \tan(A_1 + A_2 + \dots + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + \dots}{1 - S_2 + S_4 - S_6 + \dots}, \text{ where}$$

$S_1 = \tan A_1 + \tan A_2 + \dots + \tan A_n$ = the sum of the tangents of the separate angles,

$S_2 = \tan A_1 \tan A_2 + \tan A_2 \tan A_3 + \dots$ = the sum of the tangents taken two at a time,

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \dots$ = the sum of the tangents taken three at a time,
and so on.

3.2. Trigonometric Ratios of Multiple and Submultiple Angles

- (i) $\sin 2A = 2 \sin A \cos A$
 (ii) $\cos 2A = \cos^2 A - \sin^2 A$
 (iii) $\cos 2A = 2 \cos^2 A - 1$ or $1 + \cos 2A = 2 \cos^2 A$
 (iv) $\cos 2A = 1 - 2 \sin^2 A$ or $1 - \cos 2A = 2 \sin^2 A$
 (v) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$
 (vi) $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$
 (vii) $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$
 (ix) $\sin 3A = 3 \sin A - 4 \sin^3 A$
 (x) $\cos 3A = 4 \cos^3 A - 3 \cos A$
 (xi) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

3.3 Transformation Formulae

3.3A Expressing Product of Trigonometric Functions as Sum or Difference

- (i) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$ (ii) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
 (iii) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$ (iv) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

The above four formula can be obtained by expanding the right hand side and simplifying.

Note : In the fourth formula, there is a change in the pattern. Angle $(A - B)$ comes first and $(A + B)$ later. In the first quadrant, the greater the angle, the less the cosine. Hence cosine of the smaller angle is written first [to get a positive result]

3.3B Expressing Sum or Difference of Two Sines or Two Cosines as a Product

In the formulae derived in the earlier section if we put $A + B = C$ and $A - B = D$, then $A = \frac{C + D}{2}$ and $B = \frac{C - D}{2}$, these formulae can be rewritten as

$$\begin{aligned}\sin C + \sin D &= 2 \sin \frac{C + D}{2} \cdot \cos \frac{C - D}{2} \\ \sin C - \sin D &= 2 \sin \frac{C - D}{2} \cdot \cos \frac{C + D}{2} \\ \cos C + \cos D &= 2 \cos \frac{C + D}{2} \cdot \cos \frac{C - D}{2} \\ \cos C - \cos D &= -2 \sin \frac{C + D}{2} \cdot \sin \frac{C - D}{2} \quad \text{or} \quad 2 \sin \frac{C + D}{2} \cdot \sin \frac{D - C}{2}\end{aligned}$$

3.4 General formulae

- $\tan A \pm \tan B = \frac{\sin (A \pm B)}{\cos A \cos B}$ where $A, B \neq n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$
 ➤ $\cot A \pm \cot B = \frac{\sin (B \pm A)}{\sin A \sin B}$ where $A, B \neq n\pi, n \in \mathbb{Z}$

- $1 \pm \tan A \cdot \tan B = \frac{\cos(A \mp B)}{\cos A \cos B}$ where $A, B \neq n\pi + \frac{\pi}{2}, n \in \mathbb{Z}$
- $1 \pm \cot A \cdot \cot B = \pm \frac{\cos(A \mp B)}{\sin A \sin B}$ where $A, B \neq n\pi, n \in \mathbb{Z}$
- $\frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$ where $\theta \neq n\pi$
- $\frac{1 + \cos \theta}{\sin \theta} = \cot \frac{\theta}{2}$, where $\theta \neq (2n+1)\pi$
- $\frac{1 - \cos \theta}{1 + \cos \theta} = \tan^2 \frac{\theta}{2}$, where $\theta \neq (2n+1)\pi$
- $\frac{1 + \cos \theta}{1 - \cos \theta} = \cot^2 \frac{\theta}{2}$, where $\theta \neq 2n\pi$
- $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1 + \tan \theta}{1 - \tan \theta} = \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} = \frac{1 + \sin 2\theta}{\cos 2\theta}$
- $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1 - \tan \theta}{1 + \tan \theta} = \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{1 - \sin 2\theta}{\cos 2\theta}$

3.5 Values of Trigonometrical Ratios of Some Important Angles and Some Important Results

- $\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$
- $\tan 15^\circ = 2 - \sqrt{3} = \cot 75^\circ$
- $\sin 22\frac{1}{2}^\circ = \frac{1}{2}(\sqrt{2}-\sqrt{2})$
- $\tan 22\frac{1}{2}^\circ = \sqrt{2}-1$
- $\sin 18^\circ = \frac{\sqrt{5}-1}{4} = \cos 72^\circ$
- $\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54^\circ$
- $\sin 9^\circ = \frac{\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}}{4} = \cos 81^\circ$
- $\cos 36^\circ - \cos 72^\circ = \frac{1}{2}$
- $\sin \theta \sin (60^\circ - \theta) \sin (60^\circ + \theta) = \frac{1}{4} \sin 3\theta$
- $\cos \theta \cos (60^\circ - \theta) \cos (60^\circ + \theta) = \frac{1}{4} \cos 3\theta$
- $\tan \theta \tan (60^\circ - \theta) \tan (60^\circ + \theta) = \tan 3\theta$
- $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$
- $\cot 15^\circ = 2 + \sqrt{3} = \tan 75^\circ$
- $\cos 22\frac{1}{2}^\circ = \frac{1}{2}(\sqrt{2}+\sqrt{2})$
- $\cot 22\frac{1}{2}^\circ = \sqrt{2}+1$
- $\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4} = \sin 72^\circ$
- $\cos 36^\circ = \frac{\sqrt{5}+1}{4} = \sin 54^\circ$
- $\cos 9^\circ = \frac{\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}}{4} = \sin 81^\circ$
- $\cos 36^\circ \cos 72^\circ = \frac{1}{4}$

3.6 Expressions of $\sin A/2$ and $\cos A/2$ in terms of $\sin A$

$$\sqrt{1 + \sin A} = \left| \sin \frac{A}{2} + \cos \frac{A}{2} \right|$$

Note : We must be careful while determining the square root of trigonometrical function e.g.

$$\sin^2 x = |\sin x| \text{ not } \sin x$$

Illustration - 4 Show that :

$$(i) \quad \frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2}$$

$$(ii) \quad \frac{\sin \theta}{1 + \cos \theta} = \tan \frac{\theta}{2}$$

$$(iii) \quad (\cos \theta + \sin \theta)^2 = 1 + \sin 2\theta$$

$$(iv) \quad (\cos \theta - \sin \theta)^2 = 1 - \sin 2\theta$$

$$(v) \quad \tan \left(\frac{\pi}{4} + \theta \right) = \frac{1 + \tan \theta}{1 - \tan \theta} = \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}$$

$$(vi) \quad \cot \theta - \tan \theta = 2 \cot 2\theta$$

$$\tan \left(\frac{\pi}{4} - \theta \right) = \frac{1 - \tan \theta}{1 + \tan \theta} = \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta}$$

SOLUTION : (Hints)

$$(i) \quad \text{LHS} = \frac{2 \sin^2 \theta/2}{2 \sin \theta/2 \cos \theta/2} = \tan \theta/2$$

$$(ii) \quad \text{LHS} = \frac{2 \sin \theta/2 \cos \theta/2}{2 \cos^2 \theta/2} = \tan \theta/2$$

(iii) & (iv) expand LHS to get answer.

(v) expand using $\tan(A + B)$, $\tan(A - B)$ and $\tan \frac{\pi}{4} = 1$

$$(vi) \quad \text{LHS} = \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} = \frac{2 \cos 2\theta}{\sin 2\theta} = 2 \cot 2\theta$$

Illustration - 5 Calculate :

$$(i) \quad \sin 15^\circ, \cos 15^\circ, \tan 15^\circ, \cot 15^\circ$$

$$(ii) \quad \tan 22.5^\circ, \cot 22.5^\circ$$

$$(iii) \quad \tan 7.5^\circ, \cot 7.5^\circ$$

SOLUTION :

$$(i) \quad \sin 15^\circ = \sin (45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \Rightarrow \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$\cos 15^\circ = \cos (45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \Rightarrow \cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$\tan 15^\circ = \frac{1 - \cos 30^\circ}{\sin 30^\circ} \quad \left(\text{using } \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} \right)$$

$$\Rightarrow \tan 15^\circ = 2 - \sqrt{3}$$

and also, $\cot 15^\circ = \frac{1}{\tan 15^\circ} = 2 + \sqrt{3}$

(ii) $\tan 22\frac{1}{2}^\circ = \frac{1 - \cos 45^\circ}{\sin 45^\circ} = \sqrt{2} - 1$

$\cot 22\frac{1}{2}^\circ = \frac{1}{\tan 22\frac{1}{2}^\circ} = \sqrt{2} + 1$

(iii) $\tan 7\frac{1}{2}^\circ = \frac{1 - \cos 15^\circ}{\sin 15^\circ} = \frac{1 - \frac{\sqrt{3} + 1}{2\sqrt{2}}}{\frac{\sqrt{3} - 1}{2\sqrt{2}}} = \frac{2\sqrt{2} - \sqrt{3} - 1}{\sqrt{3} - 1}$

$\Rightarrow \tan 7\frac{1}{2}^\circ = (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)$

and also, $\cot 7\frac{1}{2}^\circ = \frac{1}{\tan 7\frac{1}{2}^\circ} = (\sqrt{3} + \sqrt{2})(\sqrt{2} + 1)$

Illustration - 6 Show that : $\frac{\sin 2x - \sin 3x + \sin 4x}{\cos 2x - \cos 3x + \cos 4x} = \tan 3x$

SOLUTION :

$$\begin{aligned} \text{LHS} &= \frac{\sin 2x - \sin 3x + \sin 4x}{\cos 2x - \cos 3x + \cos 4x} \\ &= \frac{(\sin 2x + \sin 4x) - \sin 3x}{(\cos 2x + \cos 4x) - \cos 3x} \quad \left[\text{note that } \frac{2x + 4x}{2} = 3x \right] \\ &= \frac{2 \sin \left(\frac{2x + 4x}{2} \right) \cos \left(\frac{2x - 4x}{2} \right) - \sin 3x}{2 \cos \left(\frac{2x + 4x}{2} \right) \cos \left(\frac{2x - 4x}{2} \right) - \cos 3x} = \frac{2 \sin 3x \cos (-x) - \sin 3x}{2 \cos 3x \cos (-x) - \cos 3x} \\ &= \frac{\sin 3x [2 \cos x - 1]}{\cos 3x [2 \cos x - 1]} \quad [\because \cos(-\theta) = \cos \theta] \\ &= \tan 3x = \text{RHS} \end{aligned}$$

Illustration - 7 Simplify : $\frac{\sin 300^\circ \tan 330^\circ \sec 420^\circ}{\tan 135^\circ \sin 210^\circ \sec 315^\circ}$

SOLUTION :

Given expression

$$\begin{aligned} &= \frac{\sin 300^\circ \tan 330^\circ \sec 420^\circ}{\tan 135^\circ \sin 210^\circ \sec 315^\circ} = \frac{\sin (360^\circ - 60^\circ) \tan (360^\circ - 30^\circ) \sec (360^\circ + 60^\circ)}{\tan (180^\circ - 45^\circ) \sin (180^\circ + 30^\circ) \sec (360^\circ - 45^\circ)} \\ &= \frac{(-\sin 60^\circ) \times (-\tan 30^\circ) \times \sec 60^\circ}{(-\tan 45^\circ) \times (-\sin 30^\circ) \times \sec 45^\circ} = \frac{\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} \times 2}{1 \times \frac{1}{2} \times \sqrt{2}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2} \end{aligned}$$

Illustration - 8

If $\cos \theta = \frac{\cos \alpha - \cos \beta}{1 - \cos \alpha \cos \beta}$, then show that $\tan \frac{\theta}{2} = \pm \tan \frac{\alpha}{2} \cot \frac{\beta}{2}$.

SOLUTION :

Manipulating the given condition as follows : $\frac{1}{\cos \theta} = \frac{1 - \cos \alpha \cos \beta}{\cos \alpha - \cos \beta}$

$$\Rightarrow \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{(1 - \cos \alpha \cos \beta) - (\cos \alpha - \cos \beta)}{(1 - \cos \alpha \cos \beta) + (\cos \alpha - \cos \beta)} \quad [\text{Apply 'C' and 'D'}]$$

$$\Rightarrow \tan^2 \frac{\theta}{2} = \frac{(1 - \cos \alpha) + \cos \beta (1 - \cos \alpha)}{(1 + \cos \alpha) - \cos \beta (1 + \cos \alpha)} = \frac{(1 - \cos \alpha)(1 + \cos \beta)}{(1 + \cos \alpha)(1 - \cos \beta)}$$

$$\Rightarrow \tan^2 \frac{\theta}{2} = \frac{1 - \cos \alpha}{1 + \cos \alpha} \cdot \frac{\tan^2 \frac{\alpha}{2}}{\tan^2 \frac{\beta}{2}} \Rightarrow \tan^2 \frac{\theta}{2} = \tan^2 \frac{\alpha}{2} \cdot \cot^2 \frac{\beta}{2} \Rightarrow \tan \frac{\theta}{2} = \pm \tan \frac{\alpha}{2} \cdot \cot \frac{\beta}{2}$$

Illustration - 9

If $\tan \frac{\theta}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{\phi}{2}$, show that $\cos \phi = \frac{\cos \theta - e}{1 - e \cos \theta}$.

SOLUTION :

We have to find $\cos \phi$ in terms of e and $\cos \theta$, so try to convert $\tan \theta/2$ to $\cos \phi$.

$$\begin{aligned} \tan^2 \frac{\theta}{2} &= \frac{1-e}{1+e} \tan^2 \frac{\phi}{2} \\ \Rightarrow \tan^2 \frac{\phi}{2} &= \frac{1+e}{1-e} \tan^2 \frac{\theta}{2} = \frac{1+e}{1-e} \left(\frac{1 - \cos \theta}{1 + \cos \theta} \right) \\ \Rightarrow \frac{\tan^2 \frac{\phi}{2}}{1} &= \frac{1+e - \cos \theta - e \cos \theta}{1 - e + \cos \theta - e \cos \theta} \\ \Rightarrow \frac{1 - \tan^2 \phi/2}{1 + \tan^2 \phi/2} &= \frac{(1 - e + \cos \theta - e \cos \theta) - (1 + e - \cos \theta - e \cos \theta)}{(1 - e + \cos \theta - e \cos \theta) + (1 + e - \cos \theta - e \cos \theta)} \quad [\text{Apply C and D}] \\ \Rightarrow \cos \phi &= \frac{-2e + 2 \cos \theta}{2 - 2e \cos \theta} = \frac{\cos \theta - e}{1 - e \cos \theta} \end{aligned}$$

Illustration - 10

If $\tan \beta = \frac{\tan \alpha + \tan \gamma}{1 + \tan \alpha \tan \gamma}$, prove that : $\sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha \sin 2\gamma}$.

SOLUTION :

We have $\tan \beta$ in terms of α and γ , so and we have to express $\sin 2\beta$ in terms of α, γ . Hence we will start with

$\sin 2\beta = \frac{2 \tan \beta}{1 + \tan^2 \beta}$ and substitute for $\tan \beta$ in R.H.S. Also, as the final expression does not contain $\tan \alpha$ and $\tan \gamma$, so express $\tan \beta$ in terms of sine and cosine.

$$\tan \beta = \frac{\sin \alpha \cos \gamma + \cos \alpha \sin \gamma}{\cos \alpha \cos \gamma + \sin \alpha \sin \gamma} = \frac{\sin (\alpha + \gamma)}{\cos (\alpha - \gamma)}$$

$$\text{Now } \sin 2\beta = \frac{2 \tan \beta}{1 + \tan^2 \beta}$$

$$\begin{aligned}
 \Rightarrow \sin 2\beta &= \frac{2 \frac{\sin(\alpha + \gamma)}{\cos(\alpha - \gamma)}}{1 + \frac{\sin^2(\alpha + \gamma)}{\cos^2(\alpha - \gamma)}} = \frac{2 \sin(\alpha + \gamma) \cos(\alpha - \gamma)}{\cos^2(\alpha - \gamma) + \sin^2(\alpha + \gamma)} \\
 &= \frac{\sin[\alpha + \gamma + \alpha - \gamma] + \sin[\alpha + \gamma - \alpha - \gamma]}{1 + \sin^2(\alpha + \gamma) - \sin^2(\alpha - \gamma)} \\
 &= \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin[\alpha + \gamma + \alpha - \gamma] \sin[\alpha + \gamma - \alpha - \gamma]} \\
 \Rightarrow \sin 2\beta &= \frac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha \sin 2\gamma}
 \end{aligned}$$

Illustration - 11 If $2 \tan \alpha = 3 \tan \beta$, then show that :

$$\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}.$$

SOLUTION :

We have to express $\tan(\alpha - \beta)$ in terms of β only. Starting with standard result of $\tan(\alpha - \beta)$ and substituting for $\tan \alpha = 3/2 \tan \beta$ in R.H.S. we have :

$$\begin{aligned}
 \Rightarrow \tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{3/2 \tan \beta - \tan \beta}{1 + 3/2 \tan^2 \beta} \\
 \Rightarrow \tan(\alpha - \beta) &= \frac{\tan \beta}{2 + 3 \tan^2 \beta} = \frac{\sin \beta \cos \beta}{2 \cos^2 \beta + 3 \sin^2 \beta} \\
 \Rightarrow &= \frac{2 \sin \beta \cos \beta}{4 \cos^2 \beta + 6 \sin^2 \beta} = \frac{\sin 2\beta}{2(1 + \cos 2\beta) + 3(1 - \cos 2\beta)} \\
 \Rightarrow \tan(\alpha - \beta) &= \frac{\sin 2\beta}{5 - \cos 2\beta}
 \end{aligned}$$

Illustration - 12 If $\alpha + \beta = 90^\circ$ and $\beta + \gamma = \alpha$ then prove that $\tan \alpha = \tan \beta + 2 \tan \gamma$.

SOLUTION : $\alpha + \beta = 90^\circ$

Taking $\tan$ on both sides

$$\text{Now } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \tan 90^\circ = \frac{\text{finite}}{0} \Rightarrow \tan \alpha \tan \beta = 1$$

$$\text{Now } \tan \alpha = \tan(\beta + \gamma) = \frac{\tan \beta + \tan \gamma}{1 - \tan \beta \tan \gamma}$$

$$\Rightarrow \tan \alpha - \tan \alpha \tan \beta \tan \gamma = \tan \beta + \tan \gamma$$

$$\Rightarrow \tan \alpha - \tan \gamma = \tan \beta + \tan \gamma \quad (\because \tan \alpha \tan \beta = 1)$$

$$\Rightarrow \tan \alpha = \tan \beta + 2 \tan \gamma$$

Illustration - 13

$a \sin^2 x + b \cos^2 x = c$, $b \sin^2 y + a \cos^2 y = d$ and $a \tan x = b \tan y$, then $\frac{a^2}{b^2}$ is

- (A) $\frac{(b-c)(d-b)}{(a-d)(c-d)}$ (B) $\frac{(a-d)(c-a)}{(b-c)(d-b)}$ (C) $\frac{(d-a)(c-a)}{(b-c)(d-b)}$ (D) $\frac{(b-c)(b-d)}{(a-c)(a-d)}$

SOLUTION : (B)

We have to find value of $\frac{a^2}{b^2}$, i.e. $\frac{\tan^2 y}{\tan^2 x}$. $[\because a \tan x = b \tan y]$

$$\text{Given that : } a \sin^2 x + b \cos^2 x = c \Rightarrow a \sin^2 x + b(1 - \sin^2 x) = c$$

$$\Rightarrow \sin^2 x(a-b) = c-b \Rightarrow \sin^2 x = \frac{c-b}{a-b}$$

$$\therefore \cos^2 x = 1 - \sin^2 x \Rightarrow \cos^2 x = \frac{a-c}{a-b}$$

$$\text{So, } \tan^2 x = \frac{c-b}{a-c} = \frac{b-c}{c-a} \dots \dots \text{(i)}$$

Similarly, we can find $\tan^2 y$

$$\Rightarrow \tan^2 y = \frac{a-d}{d-b} \quad [\text{Replace } c \text{ by } d, b \text{ by } a, a \text{ by } b \text{ in (i)}]$$

$$\Rightarrow \frac{a^2}{b^2} = \frac{\tan^2 y}{\tan^2 x} = \frac{a-d}{d-b} \times \frac{c-a}{b-c}$$

Illustration - 14

If $x = \sin(\alpha - \beta) \cdot \sin(\gamma - \delta)$; $y = \sin(\beta - \gamma) \cdot \sin(\alpha - \delta)$ and $z = \sin(\gamma - \alpha) \cdot \sin(\beta - \delta)$, then

- (A) $x + y + z = 0$ (B) $x + y - z = 0$ (C) $y + z - x = 0$ (D) None of these

SOLUTION : (A)

Consider $x = \sin(\alpha - \beta) \sin(\gamma - \delta)$, multiply both side by '2'. We get :

$$\therefore 2x = 2 \sin(\alpha - \beta) \sin(\gamma - \delta)$$

$$2x = \cos(\alpha - \beta - \gamma + \delta) - \cos(\alpha - \beta + \gamma - \delta) \dots \dots \text{(i)}$$

And similarly

$$2y = \cos(\beta - \gamma - \alpha + \delta) - \cos(\beta - \gamma + \alpha - \delta) \dots \dots \text{(ii)}$$

$$2z = \cos(\gamma - \alpha - \beta + \delta) - \cos(\gamma - \alpha + \beta - \delta) \dots \dots \text{(iii)}$$

Now adding at (i), (ii) and (iii) to get :

$$2x + 2y + 2z = \cos(\alpha - \beta - \gamma + \delta) - \cos(\alpha - \beta + \gamma - \delta) + \cos(\beta - \gamma - \alpha + \delta) - \cos(\beta - \gamma + \alpha - \delta) \\ + \cos(\gamma - \alpha - \beta + \delta) - \cos(\gamma - \alpha + \beta - \delta)$$

$$\text{As } \cos(\alpha - \beta - \gamma + \delta) = \cos(\gamma - \alpha + \beta - \delta)$$

$$\cos(\beta - \gamma - \alpha + \delta) = \cos(\alpha - \beta + \gamma - \delta)$$

$$\cos(\gamma - \alpha - \beta + \delta) = \cos(\beta - \gamma - \alpha - \delta)$$

$$[\text{As } \cos(-\theta) = \cos \theta]$$

$$\Rightarrow 2x + 2y + 2z = 0 \Rightarrow x + y + z = 0$$

Illustration - 15 The value of $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$ is :

- (A) 1 (B) -1 (C) 0 (D) 4

SOLUTION : (D)

$$\text{Consider } \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sqrt{3} \cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cdot \cos 20^\circ}$$

Multiply and divided by

$$\begin{aligned} &= \frac{4 \left(\frac{\sqrt{3}}{2} \cos 20^\circ - \frac{1}{2} \sin 20^\circ \right)}{2 \sin 20^\circ \cos 20^\circ} \\ &= 4 \cdot \frac{(\sin 60^\circ \cdot \cos 20^\circ - \cos 60^\circ \cdot \sin 20^\circ)}{\sin 40^\circ} = 4 \cdot \frac{\sin (60^\circ - 20^\circ)}{\sin 40^\circ} = 4 \cdot \frac{\sin 40^\circ}{\sin 40^\circ} = 4 \end{aligned}$$

Illustration - 16 The value of expression $\tan A + 2 \tan 2A + 4 \tan 4A + 8 \cot 8A$ is :

- (A) $\tan A$ (B) $\cot A$ (C) $\tan \frac{A}{2}$ (D) $\cot \frac{A}{2}$

SOLUTION : (B)

$$\begin{aligned} &\tan A + 2 \tan 2A + 4 \tan 4A + 8 \left(\frac{1 - \tan^2 4A}{2 \tan 4A} \right) \quad \left[\text{By using } \cot 2\theta = \left(\frac{1 - \tan^2 \theta}{2 \tan \theta} \right) \right] \\ &= \tan A + 2 \tan 2A + \left(\frac{4 \tan^2 4A + 4 - 4 \tan^2 4A}{\tan 4A} \right) = \tan A + 2 \tan 2A + 4 \cot 4A \\ &= \tan A + 2 \tan 2A + 4 \left(\frac{1 - \tan^2 2A}{2 \tan 2A} \right) \quad \left[\text{By using } \cot 2\theta = \left(\frac{1 - \tan^2 \theta}{2 \tan \theta} \right) \right] \\ &= \tan A + \left[\frac{2 \tan^2 2A + 2 - 2 \tan^2 2A}{\tan 2A} \right] = \tan A + 2 \cot 2A \\ &= \tan A + 2 \left(\frac{1 - \tan^2 A}{2 \tan A} \right) \quad \left[\text{By using } \cot 2\theta = \left(\frac{1 - \tan^2 \theta}{2 \tan \theta} \right) \right] \\ &= \frac{\tan^2 A + 1 - \tan^2 A}{\tan A} = \cot A \end{aligned}$$

Note : Students are advised to learn above result as formula.

$$\text{i.e., } \tan A + 2 \cot 2A = \cot A$$

Illustration - 17 Find set of all possible values of α in $[-\pi, \pi]$ such that $\sqrt{\frac{1 - \sin \alpha}{1 + \sin \alpha}}$ is equal to $(\sec \alpha - \tan \alpha)$.

- (A) $0 < \alpha < \frac{\pi}{3}$ (B) $-\pi < \alpha < \frac{\pi}{4}$ (C) $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ (D) $-\pi < \alpha < \pi$

SOLUTION : (C)

Clearly $\alpha \neq \pm \frac{\pi}{2}$

as, $\sec \alpha - \tan \alpha = \frac{1 - \sin \alpha}{\cos \alpha} \quad \dots (i)$

and $\sqrt{\frac{1 - \sin \alpha}{1 + \sin \alpha}} = \sqrt{\frac{(1 - \sin \alpha)^2}{\cos^2 \alpha}} = \left| \frac{1 - \sin \alpha}{\cos \alpha} \right| = \frac{1 - \sin \alpha}{|\cos \alpha|} \quad \dots (ii) \quad \text{As } [1 - \sin \alpha \text{ is always +ve}]$

From (i) and (ii) two expressions are equal only if $\cos \alpha > 0$, i.e., $-\pi/2 < \alpha < \pi/2$

$\therefore \sqrt{\frac{1 - \sin \alpha}{1 + \sin \alpha}}$ and $\sec \alpha - \tan \alpha$ are equal only where $\alpha = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Illustration - 18

If $\frac{x}{\cos \theta} = \frac{y}{\cos\left(\theta + \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta - \frac{2\pi}{3}\right)}$, then $x + y + z$ is equal to :

- (A) -1 (B) 1 (C) 0 (D) None of these

SOLUTION : (C)

Given $\frac{x}{\cos \theta} = \frac{y}{\cos\left(\theta + \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \lambda$ (say)

$\Rightarrow x + y + z = \lambda \left\{ \cos \theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta - \frac{2\pi}{3}\right) \right\} = \lambda \left\{ \cos \theta + 2 \cos \theta \cos \frac{2\pi}{3} \right\} = 0$

Illustration - 19 Match the following :**Column I**

(A) $\cos^2 52 \frac{1}{2}^\circ - \sin^2 22 \frac{1}{2}^\circ$

(B) $\cos^2 \frac{3\pi}{5} + \cos^2 \frac{4\pi}{5}$

(C) $\sin 24^\circ + \cos 6^\circ$

(D) $\sin^2 50^\circ + \cos^2 130^\circ$

Column II

(P) 1

(Q) $\frac{3 - \sqrt{3}}{4\sqrt{2}}$

(R) $\frac{3}{4}$

(S) $\frac{\sqrt{15} + \sqrt{3}}{4}$

SOLUTION : [A-Q] [B-R] [C-S] [D-P]

(i) $\cos^2 52 \frac{1}{2}^\circ - \sin^2 22 \frac{1}{2}^\circ = \cos 75^\circ \cos 30^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}} \times \frac{\sqrt{3}}{2} = \frac{3 - \sqrt{3}}{4\sqrt{2}}$

(ii) $\cos^2 \frac{3\pi}{5} + \cos^2 \frac{4\pi}{5} = \left(\frac{\sqrt{5} - 1}{4}\right)^2 + \left(\frac{\sqrt{5} + 1}{4}\right)^2 = \frac{3}{4}$

(iii) $\sin 24^\circ + \cos 6^\circ = 2 \sin 54^\circ \cos 30^\circ = \frac{\sqrt{15} + \sqrt{3}}{4}$

(iv) $\sin^2 50^\circ + \cos^2 130^\circ = 1$